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THE DEFINITION OF THE DEFINITE INTEGRAL IN TERMS OF DARBOUX SUMS AND THE CONDITIONS FOR INTEGRABILITY

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Abstract: This paper presents a rigorous development of the definite integral through the framework of Darboux sums, emphasizing both theoretical foundations and practical implications. The concepts of upper and lower Darboux sums are introduced as fundamental tools for approximating the area under a bounded function over a closed interval. By examining partitions of the interval and the behavior of supremum and infimum values on subintervals, the study establishes the formal definition of the definite integral. A key focus is placed on the necessary and sufficient conditions for integrability, particularly the convergence of upper and lower sums to a common limit.

Keywords: Definite Integral, Darboux Sums, Upper Sum, Lower Sum, Integrability, Partition of an Interval, Supremum and Infimum, Bounded Functions, Riemann Integral, Real Analysis.

Introduction: The concept of the definite integral occupies a central position in mathematical analysis, serving as a fundamental tool for measuring accumulated quantities such as area, volume, and total change. Its development is closely connected with the need to rigorously formalize the intuitive idea of summing infinitely many infinitesimal contributions. Among the various approaches to defining the definite integral, the method based on Darboux sums provides a clear and logically consistent framework that emphasizes the role of bounds and limits in the construction of the integral [1].

Darboux sums, introduced as upper and lower sums, allow for the approximation of the area under a bounded function over a closed interval by partitioning the interval into smaller subintervals. On each subinterval, the supremum and infimum of the function are used to construct rectangles that either overestimate or underestimate the actual area. By refining the partition, these approximations become increasingly accurate, leading to a natural definition of the definite integral as the common limit of the upper and lower sums, provided such a limit exists [2].

A crucial aspect of this approach is the notion of integrability. A function is said to be Darboux integrable if the difference between its upper and lower sums can be made arbitrarily small by choosing sufficiently fine partitions. This condition ensures that both sums converge to the same value, which is then defined as the definite integral. The study of integrability not only establishes the existence of the integral but also reveals important connections with other properties of functions, such as continuity, monotonicity, and boundedness [3].

Furthermore, the Darboux approach is closely related to the Riemann integral, and it can be shown that both definitions are equivalent for a wide class of functions. This equivalence highlights the robustness of the concept of integration and its consistency across different formulations. In this paper, we explore the definition of the definite integral via Darboux sums, examine the conditions for integrability, and illustrate the theory with relevant examples, thereby providing a comprehensive understanding of this essential topic in real analysis [4].

Theoretical Background:The rigorous formulation of the definite integral requires several fundamental concepts from real analysis, including partitions of intervals, bounded functions, and the notions of supremum and infimum. Let $f:[a,b] \to R$ be a real-valued function defined on a closed and bounded interval [a,b]. A **partition** P of the interval [a,b] is defined as a finite ordered set of points

$$P=\{x_0, x_1, x_2, \quad , x_n\}$$

such that

$$a=x_0<x_1<x_2<\cdots<x_n=b$$

Each subinterval $[x_{i-1}, x_i]$ is associated with the function's behavior over that segment.

A function f is said to be **bounded** on [a,b] if there exist real numbers m and M such that $m \le f(x) \le M$ for all $x \in [a,b]$.

For each subinterval $[x_{i-1}, x_i]$, one defines the **infimum** $m_i = \inf\{f(x): x \in [x_{i-1}, x_i]\}$ and the **supremum** $M_i = \sup\{f(x): x \in [x_{i-1}, x_i]\}$. These values represent the greatest lower bound and least upper bound of the function on the given subinterval, respectively. Using these quantities, the **Darboux lower sum** and **upper sum** of the function with respect to the partition P are defined as

$$L(f,P)=\sum_{i=1}^{n} m_i(x_i-x_{i-1}), \quad U(f,P)=\sum_{i=1}^{n} M_i(x_i-x_{i-1}).$$

The lower sum provides an underestimation, while the upper sum provides an overestimation of the area under the graph of the function over $[a,b]$

An essential property of these sums is that for any partition P, the inequality

$L(f,P) \le U(f,P)$

always holds. Moreover, as the partition becomes finer (i.e., the maximum length of subintervals tends to zero), the lower sums increase and the upper sums decrease. This monotonic behavior forms the basis for defining the definite integral and establishes the groundwork for the concept of integrability in the sense of Darboux. **Definition of the Definite Integral via Darboux Sums:**Building upon the foundational concepts of partitions, boundedness, and Darboux sums, the definite integral can be rigorously defined using the limiting behavior of upper and lower sums. Let $f:[a,b] \to R$ bounded function defined on a closed interval $[a,b]$. Consider the collection of all possible partitions P of the interval. For each partition, the corresponding lower sum $L(f,P)$ and upper sum $U(f,P)$ are constructed using the infimum and supremum of the function over each subinterval.

The set of all lower sums is bounded above, while the set of all upper sums is bounded below. Therefore, one can define the **lower Darboux integral** as

$$\underline{\int_a^b} f(x)dx=\sup\{L(f,P):P\} \text{ is a partition of [a,b],}$$

and the **upper Darboux integral** as

$$\overline{\int_a^b} f(x)dx=\inf\{U(f,P):P \text{ is a partition of [a,b]}\}.$$

These two quantities represent, respectively, the greatest possible lower approximation and the least possible upper approximation of the area under the graph of the function. The fundamental idea behind integration in the Darboux sense is that, if these two values coincide, then the function admits a well-defined integral.

More precisely, the function fff is said to be **Darboux integrable** on $[a,b]$ if

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$$

In this case, the common value is called the **definite integral** of f over $[a,b]$, and is denoted by

$$\int_a^b f(x)dx.$$

An equivalent and often more intuitive characterization of integrability is given in terms of the difference between upper and lower sums. A bounded function f is Darboux integrable if and only if for every $\varepsilon>0$, there exists a partition P such that

$U(f,P)-L(f,P)<\varepsilon$.

This condition expresses the idea that the upper and lower approximations can be made arbitrarily close by refining the partition.

Thus, the Darboux definition of the definite integral provides a precise and logically consistent method for capturing the notion of accumulated quantity, forming a cornerstone of modern real analysis.

Conditions for Integrability:A central problem in the theory of the definite integral is to determine under what conditions a bounded function is Darboux integrable. As established in the previous section, a function $f:[a,b]\to R$ is integrable in the sense of Darboux if and only if its lower and upper Darboux integrals coincide. This fundamental criterion can be reformulated in several equivalent ways, each offering valuable insight into the structure of integrable functions.

One of the most important characterizations states that a bounded function f is Darboux integrable on $[a,b]$ if for every $\varepsilon>0$, there exists a partition P such that

$U(f,P)-L(f,P)<\varepsilon$.

This condition reflects the idea that the discrepancy between the upper and lower approximations of the area under the curve can be made arbitrarily small. In other words, the oscillation of the function over sufficiently small subintervals must diminish in a controlled manner.

A key concept related to integrability is the **oscillation** of a function on a subinterval. For each subinterval $[x_{i-1},x_i]$, the oscillation is defined as

$\omega_i=M_i-m_i$,

where M_i and m_i are the supremum and infimum of f on that subinterval. The difference between the upper and lower sums can then be expressed as

$$U(f,P)-L(f,P)=\sum_{i=1}^{n} w_i(x_i-x_{i-1}).$$

Thus, integrability requires that the total weighted oscillation tends to zero as the partition is refined.

Several important classes of functions are known to be Darboux integrable. First, every **continuous function** on a closed interval $[a,b]$ is integrable. This follows from the fact that continuous functions on closed intervals are uniformly continuous, ensuring that their oscillation on sufficiently small subintervals can be made arbitrarily small. Consequently, the difference between upper and lower sums can be controlled effectively.

Second, every **monotonic function** on $[a,b]$ is also integrable. Even though such functions may have jump discontinuities, these discontinuities are limited in number and do not prevent the

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convergence of Darboux sums. The monotonicity ensures that the supremum and infimum on each subinterval are attained at the endpoints, simplifying the analysis.

More generally, a bounded function is Darboux integrable if the set of its discontinuities has **measure zero**. This important result connects the theory of integration with measure theory and provides a broad criterion for integrability. It implies that functions with a finite or countable number of discontinuities are integrable, while functions with “too many” discontinuities, such as the Dirichlet function, fail to be integrable.

These conditions collectively demonstrate that Darboux integrability is a robust and flexible concept, capable of accommodating a wide variety of functions while maintaining a precise mathematical structure

Relationship Between Darboux and Riemann Integration: The Darboux approach to integration is closely connected with the classical Riemann integral, and one of the fundamental results in real analysis is that these two definitions are equivalent for bounded functions on closed intervals. Although they originate from different constructions, both approaches aim to formalize the same intuitive idea: the computation of the area under a curve through limiting processes of finite approximations.

In the Riemann framework, integration is defined using tagged partitions and Riemann sums of the form

$$S(f,P,\xi)=\sum_{i=1}^{n} f(\xi_i)(x_i-x_{i-1}),$$

where $\xi_i \in [x_{i-1}, x_i]$.The integral exists if these sums converge to a common limit as the mesh of the partition tends to zero. In contrast, the Darboux approach avoids arbitrary sample points and instead relies on extremal values (supremum and infimum) over each subinterval, producing upper and lower sums that bound all possible Riemann sums.

A key result establishing equivalence is that a bounded function is Riemann integrable if and only if it is Darboux integrable. Moreover, when the integral exists, all three quantities coincide:

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

This equivalence shows that both theories describe the same limiting process, but from slightly different perspectives: Riemann integration emphasizes sampling points, while Darboux integration emphasizes bounding behavior. One advantage of the Darboux formulation is its conceptual clarity in establishing integrability conditions. Since upper and lower sums depend only on supremum and infimum values, it becomes easier to analyze convergence without selecting specific evaluation points. This makes Darboux integration particularly suitable for theoretical investigations in real analysis.

Another important implication of this equivalence is that many classical results in integration theory can be proven using either approach. For instance, properties such as linearity, additivity over intervals, and order preservation hold in both frameworks. Additionally, the Fundamental Theorem of Calculus can be developed consistently once integrability is established.

The equivalence also highlights that discontinuities play a crucial role in determining integrability. Both Darboux and Riemann theories show that a bounded function is integrable if its discontinuities are sufficiently “small” in a measure-theoretic sense. This connection later leads to more advanced integration theories, such as Lebesgue integration, which generalize these ideas further. Thus, the Darboux and Riemann integrals are not competing definitions but rather two equivalent formulations of the same mathematical concept, each offering unique insights into the structure of integration.

Illustrative Examples of Darboux Integrability: To clarify the theoretical concepts introduced in the previous sections, we consider several illustrative examples that demonstrate the computation of Darboux upper and lower sums, as well as the verification of integrability.

Example 1: Linear Function Let $f(x)=x$ on the interval [0,1]. Since f is continuous on a closed interval, it is Darboux integrable. For any partition P, the function is increasing, hence the infimum and supremum on each subinterval occur at the endpoints. As the mesh of the partition tends to zero, the difference $U(f,P)-L(f,P)$ approaches zero. Consequently,

$$\int_0^1 x\,dx = \frac{1}{2}.$$

Example 2: Quadratic Function: Consider $f(x)=x^2$ on [0,1]. The function is continuous and monotonic increasing, which guarantees integrability. By constructing equal partitions and evaluating upper and lower sums, one observes convergence to a common limit. The exact value of the integral is

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

Example 3: Step Discontinuity Function: Let

$$f(x) = \begin{cases} 1, & \frac{1}{2} \le x, \\ 0, & x < \frac{1}{2} \end{cases}.$$

This function has a single jump discontinuity at x=1/2. Despite this discontinuity, the function is still Darboux integrable because the set of discontinuities has measure zero. The upper and lower sums converge to the same value, giving

$$\int_0^1 f(x)\,dx = \frac{1}{2}.$$

These examples demonstrate that continuity is sufficient but not necessary for integrability. Even functions with finitely many discontinuities remain integrable under the Darboux framework, highlighting the robustness of the theory.

Conclusion: The study of the definite integral through the framework of Darboux sums provides a rigorous and conceptually clear foundation for one of the most fundamental notions in mathematical analysis. By introducing upper and lower sums based on the supremum and infimum of a bounded function over subintervals, the Darboux approach replaces intuitive geometric ideas with precise analytical constructions. This allows the notion of "area under a curve" to be defined in a logically consistent and mathematically sound manner. Furthermore, the equivalence between Darboux and Riemann integrals shows that both theories describe the same mathematical object from different perspectives. While the Riemann approach emphasizes sample points and summation processes, the Darboux method focuses on bounding behavior through extremal values. This duality not only strengthens the theoretical foundation of integration but also enhances its applicability in various branches of mathematics. In conclusion, the Darboux formulation of the definite integral offers both theoretical elegance and practical utility. It plays a crucial role in real analysis by providing clear criteria for integrability and by establishing a solid bridge between intuitive geometric ideas and rigorous mathematical reasoning.

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