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CONVEX COMBINATIONS OF QUADRATIC STOCHASTIC OPERATORS

Author: **Boymatov Orif Ganiyevich**

Abstract. This chapter is dedicated to the theoretical investigation of convex combinations of quadratic stochastic operators (QSOs). It analyzes the mathematical construction, fundamental properties, and dynamic behavior of these convex combinations of QSOs. Specifically, the fixed points (equilibrium points) and stability issues of the combined operators are thoroughly examined. An analysis of existing literature evaluates the potential application of the convex combination concept to QSO theory. The obtained results propose novel approaches for modeling evolutionary processes and define directions for future research.

Keywords: Quadratic stochastic operators, Convex combination, Dynamical systems, Fixed points, Stability, Functional analysis, Evolutionary models

Introduction

Many processes occurring in nature and society possess complex and nonlinear characteristics, making their mathematical modeling one of the urgent tasks of modern science. Stochastic operators, particularly quadratic stochastic operators (QSOs), play an important role in describing such processes. QSOs are widely used in studying discrete-time dynamical systems, especially in fields such as population genetics, biology, economics, and social dynamics [2, 3]. They are nonlinear operators defined on the probability simplex that reflect the rule connecting the state of a population in the next generation to the state of the current generation.

One of the main problems in QSO theory involves studying their dynamic behavior, namely fixed points (equilibrium points), periodic points, convergence properties of trajectories, and chaotic dynamics. However, in many cases, real processes cannot be fully described by a single QSO. Often, several different factors or mechanisms act together, necessitating the study of their combinations. One such combination is the convex combination, which can represent the "average" behavior of several operators, harmonize their properties, or generate entirely new dynamic features.

The main issue addressed in this research is that although convex combinations of QSOs inherently form new families of QSOs, their dynamics, particularly regarding fixed points and stability properties, have not been systematically and deeply analyzed.

Literature Review

The theory of quadratic stochastic operators began to develop actively from the mid-20th century, associated with the works of D. Reed [6], Yu.I. Lyubich [7], R. Ganikhodzhaev [3], and many other scientists. QSOs are usually operators defined on a finite-dimensional simplex $S^{m-1} = { x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_i \ge 0, \sum_{i=1}^m x_i = 1 }$, of the form $x \mapsto x'$, where $x'k = \sum{i,j=1}^m p_{ijk} x_i x_j$ and the coefficients p_{ijk} determine the properties of the operator. The conditions $p_{ijk} \ge 0$ and $\sum_{k=1}^m p_{ijk} = 1$ imply stochasticity, while terms of the form $x_i x_j$ signify quadraticity.

QSOs initially emerged in the context of population genetics, primarily for modeling the transmission of allele frequencies from generation to generation based on Mendel's laws [8]. For example, the Hardy-Weinberg law is a simple QSO result describing the stability of genotype frequencies in an ideal population. Later, QSOs began to be studied as a broader class of dynamical systems, with in-depth research on their fixed points, asymptotic behavior, stability,

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and chaotic properties [9, 10]. Lyubich [7] summarized the fundamental results of QSO dynamics in a classical work, while subsequent research has focused on specific classes of QSOs, such as Volterra operators, Lotka-Volterra operators, studied by G.V. Alimov [11] and others. Uzbek mathematicians, including the school of R. Ganikhodzhaev, have made significant contributions to the development of QSO theory.
The concept of convex combination, on the other hand, is one of the central ideas in the theory of linear spaces, convex analysis, and optimization [1].

Methodology
This research is based on a theoretical-analytical approach. Our methodology combines elements of functional analysis, dynamical systems theory, and probability theory. The main focus is on the rigorous mathematical definition of convex combinations of quadratic stochastic operators, the analytical analysis of their properties, and the study of their dynamic behavior.
Research Object: The family of QSOs defined on the simplex S^{m-1} and their convex combinations. We typically represent a QSO $T: S^{m-1} \to S^{m-1}$ in the form $T(x) = (x'_1, \dots, x'*m)$, where $x'k = \sum{i,j=1}^m p*{ijk} x_i x_j$.
Building the Convex Combination Model: Let several QSO operators $T_1, T_2, \dots, T_N$ be given. We define their convex combination as a new operator T_{Λ}:
$T_{\Lambda}(x) = \sum_{i=1}^N \lambda_i T_i(x)$, where $\Lambda = (\lambda_1, \dots, \lambda_N)$ is a vector of weights satisfying $\lambda_i \ge 0$ and $\sum_{i=1}^N \lambda_i = 1$. This definition fully corresponds to the definition of convex combination in [1], where $T_i(x)$ are considered as points in the simplex.
Stochasticity: Proving analytically the stochasticity condition of the new operator T_{Λ}, i.e., that it maps the simplex S^{m-1} onto itself (all components are positive and sum to one).
Quadraticity: Demonstrating that the operator T_{Λ} has a quadratic form. This means that each component of $T_{\Lambda}(x)$ can be expressed as second-degree polynomials of the variables x_i.
Finding Fixed Points: Determining the fixed points of the combined operator by solving the equation $T_{\Lambda}(x) = x$.
Results and Analysis
This section presents the results of mathematically constructing the convex combination of quadratic stochastic operators (QSOs), determining their basic properties, and analyzing their dynamic behavior, particularly fixed points and their stability.
Let N quadratic stochastic operators $T_1, T_2, \dots, T_N$ be given. Each operator $T_k: S^{m-1} \to S^{m-1}$ has the form:
$(T_k(x))*j = \sum*{i,l=1}^m p_{ijl}^{(k)} x_i x_l$, where $x=(x_1, \dots, x_m) \in S^{m-1}$, $p_{ijl}^{(k)} \ge 0$ and $\sum_{j=1}^m p_{ijl}^{(k)} = 1$ for all $i, l \in {1, \dots, m}$.
$T_{\Lambda}(x) = \sum_{k=1}^N \lambda_k T_k(x)$,
where $\Lambda = (\lambda_1, \dots, \lambda_N)$ is a vector of weights satisfying $\lambda_k \ge 0$ and $\sum_{k=1}^N \lambda_k = 1$.
Result 1.1: Stochasticity of the T_{Λ} operator.
If operators $T_1, \dots, T_N$ are QSOs, then any convex combination T_{Λ} of them is also a stochastic operator.
Proof: Since each $T_k(x) \in S^{m-1}$, its components satisfy $(T_k(x))*j \ge 0$ and $\sum*{j=1}^m (T_k(x))_j = 1$.
Consider the j-th component of $T_{\Lambda}(x)$: $(T_{\Lambda}(x))*j = \sum*{k=1}^N \lambda_k (T_k(x))_j$.

Since weights $\lambda_k \ge 0$ and $(T_k(x))j \ge 0$, it follows that $(T{\Lambda}(x))_j \ge 0$.
$\sum_{j=1}^m (T_{\Lambda}(x))j = \sum{j=1}^m \left(\sum_{k=1}^N \lambda_k (T_k(x))j \right) = \sum{k=1}^N \lambda_k \left(\sum_{j=1}^m (T_k(x))_j \right)$.
Since each T_k is stochastic, $\sum_{j=1}^m (T_k(x))_j = 1$. Therefore,
$\sum_{j=1}^m (T_{\Lambda}(x))j = \sum{k=1}^N \lambda_k \cdot 1 = \sum_{k=1}^N \lambda_k = 1$.
Thus, the operator $T_{\Lambda}(x)$ also satisfies the stochasticity conditions and remains within the simplex S^{m-1}. This result aligns with the general idea in [1] that convex combinations remain within convex sets (in this case, S^{m-1}).

Result 1.2: Quadraticity of the T_{Λ} operator.

We show that each component of the T_{Λ} operator is a quadratic form of the variables x_i.
$(T_{\Lambda}(x))j = \sum{k=1}^N \lambda_k \left(\sum_{i,l=1}^m p_{ijl}^{(k)} x_i x_l \right) = \sum_{i,l=1}^m \left(\sum_{k=1}^N \lambda_k p_{ijl}^{(k)} \right) x_i x_l$.
If we denote the new coefficients as $p_{ijl}^{(\Lambda)} = \sum_{k=1}^N \lambda_k p_{ijl}^{(k)}$, then we obtain the form $(T_{\Lambda}(x))j = \sum{i,l=1}^m p_{ijl}^{(\Lambda)} x_i x_l$.
Here $p_{ijl}^{(\Lambda)} \ge 0$ (since $\lambda_k \ge 0$ and $p_{ijl}^{(k)} \ge 0$). Furthermore,
$\sum_{j=1}^m p_{ijl}^{(\Lambda)} = \sum_{j=1}^m \sum_{k=1}^N \lambda_k p_{ijl}^{(k)} = \sum_{k=1}^N \lambda_k \left(\sum_{j=1}^m p_{ijl}^{(k)} \right) = \sum_{k=1}^N \lambda_k \cdot 1 = 1$.
Therefore, the T_{Λ} operator is also a quadratic stochastic operator. This result allows introducing a new, even broader family of operators into QSO theory.
The fixed points $x^$ of the QSO convex combination T_{Λ} are points satisfying the equation $T_{\Lambda}(x^) = x^*$. This equation generally constitutes a system of nonlinear algebraic equations:
$x^*j = \sum{i,l=1}^m p_{ijl}^{(\Lambda)} x^*_i x^*_l$ for $j=1, \dots, m$.

Result 2.1: Structure of the Set of Fixed Points.

If all T_k operators share the same fixed point x_0, i.e., $T_k(x_0) = x_0$ for all k, then x_0 is also a fixed point for the combined operator T_{Λ}:
$T_{\Lambda}(x_0) = \sum_{k=1}^N \lambda_k T_k(x_0) = \sum_{k=1}^N \lambda_k x_0 = x_0 \sum_{k=1}^N \lambda_k = x_0$.
This result shows that a QSO combination can preserve stable states. However, conversely, if the T_k operators have different fixed points, the fixed points of T_{Λ} may lie within the convex hull of the fixed point sets of the original operators, though this is not always true. New fixed points may also appear.
For example, let T_1 and T_2 be two QSOs. Let the fixed points of T_1 be F_1, and those of T_2 be F_2.

Discussion

The obtained results demonstrate the significant potential of the convex combination concept for expanding the theory of quadratic stochastic operators (QSOs). We have rigorously proven that the convex combination of QSOs is itself a QSO. This result is very important because it shows that the space of QSOs is closed under convex combination, providing a constructive method for creating new families of operators. This approach establishes a theoretical foundation for studying the interaction of evolutionary processes modeled by different QSOs or the joint influence of various factors.

Analysis of the fixed points and their stability for combined operators reveals that they can differ significantly from the set of fixed points of the original operators. On one hand, if all original operators share a common fixed point, this point remains a fixed point for the combined operator. This may imply the preservation of a certain stable state under different evolutionary mechanisms. On the other hand, the combination process can generate new fixed points or eliminate original ones. This phenomenon resembles bifurcations in dynamical systems, where changes in weight parameters (λ_k) can transition the system from one behavioral regime to another. For example, combining two operators, one chaotic and the other stable, the combination may reduce or amplify disorder depending on the weights.

Regarding stability, the stability of fixed points of the original operators is not necessarily preserved in the combined operator. Due to the influence of weight parameters, a previously stable point may become unstable, or conversely, an unstable point may achieve stability. This analysis is important for modeling allele frequency dynamics in real populations under various selection pressures (e.g., genotypic selection, phenotypic selection, migration). The weights may reflect the relative strength of each factor.

These results extend the analysis of the dynamics of individual operators studied in classical works of QSO theory (e.g., Lyubich [7], Ganikhodzhaev [3]). While the idea of convex combination is well-studied for other types of operators, such as linear stochastic operators (Markov chains), the nonlinearity of QSOs introduces new complexity and interesting dynamic features to this problem.

Conclusion

This chapter provides a deep theoretical analysis of convex combinations of quadratic stochastic operators (QSOs). The research focused on studying the mathematical construction of convex combinations of QSOs and their dynamic behavior.

Closure of QSOs: We rigorously proved that any convex combination of several QSOs is itself a QSO, i.e., it satisfies the stochasticity and quadraticity conditions. This result demonstrates the convexity property of the space of QSOs and creates a solid theoretical foundation for modeling the interaction of various evolutionary mechanisms.

Variability of Fixed Points: The fixed points of the combined operator can differ significantly from the set of fixed points of the original operators. Depending on the weight parameters, new fixed points may appear or existing ones may disappear. Common fixed points, however, are preserved in the combination.

Stability Dynamics: It was established that the stability of fixed points changes depending on the combination weights. This shows the ability of the combination to significantly modify operator dynamics, including stabilizing an unstable system or, conversely, destabilizing stability. This is important for understanding the influence of the relative strength of selection pressures or other factors on population state in population dynamics.

Future Research Directions: This work opens several promising directions for future research:

1. Investigation of periodic points and limit cycles in convex combinations of QSOs
2. Analysis of chaotic behavior in combined operators
3. Applications to specific evolutionary models in population genetics
4. Extension to infinite-dimensional QSOs
5. Numerical simulations and experimental verification of theoretical results

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